MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2014/15
– Blatt 3 –
29.10.2014

# PDG I
# (Zentralübung)

## Problem Sheet 3

### Question 1

Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g \, \mathrm{d}S + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f \, \mathrm{d}x \,,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B(0,r) \\ u = g & \text{on } \partial B(0,r) \,. \end{cases}$$

### Question 2

Suppose $u \in C^\infty(\mathbb{R}^n)$ satisfies $\Delta u = 0$. Let $O$ be an orthogonal $n \times n$ matrix, and define

$$v(x) := u(Ox) \,, \qquad (x \in \mathbb{R}^n) \,.$$

Show that $\Delta v = 0$ too (so Laplace's equation is rotation invariant).

### Question 3

Suppose $u : x \in \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ has the form $u(x) = f(|x|)$, where $f \in C^2(0,\infty)$ (so $u$ is rotationally invariant). Show that

$$\Delta u(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|) \,.$$

Use this to determine all rotationally invariant *harmonic* maps $u$ on $\mathbb{R}^n \setminus \{0\}$.

**Deadline for handing in: 0800 Wednesday 5 November**

*Please put solutions in Box 17, 1st floor (near the library)*

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`

# Sheet 3

① Suppose $n \geq 3$ and $u \in C^\infty(\overline{B(0,r)})$ satisfies

$$(1) \quad \begin{cases} -\Delta u(x) = \cancel{g(x)}\, f(x), & x \in B(0,r) \\ u(x) = g(x), & x \in \partial B(0,r) \end{cases}$$

$f : B(0,r) \to \mathbb{R}$, $g : \partial B(0,r) \to \mathbb{R}$ smooth

Show:

$$u(0) = \fint_{\partial B(0,r)} g(x)\, dS(x) + \frac{1}{n(n-2)\alpha_n} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x)\, dx$$

Like in lectures, for $s \in (0,r]$, define

$$\varphi(s) := \fint_{\partial B(0,s)} u(x)\, dS(x) = \fint_{\partial B(0,1)} u(sz)\, dS(z)$$

Then

$$\varphi'(s) = \fint_{\partial B(0,1)} \nabla u(sz) \cdot z \, dS(z)$$

normal vector to $\partial B(0,s)$

$$= \fint_{\partial B(0,s)} \nabla u(\cancel{sz}\, x) \cdot \frac{x}{s} \, \cancel{dS(z)}\, dS(x)$$

div thm

$$= \frac{1}{s^{n-1}\omega_n} \int_{B(0,s)} \Delta u(x)\, \cancel{dS(x)}\, dx$$

(1)

$$= \frac{1}{s^{n-1}\omega_n} \int_{B(0,s)} f(x)\, dx = \frac{1}{s^{n-1} n\alpha_n} \int_{B(0,s)} f(x)\, dx$$

By (1), $\lim_{s \uparrow r} \varphi(s) = \fint_{\partial B(0,r)} g(x)\, dS(x)$

Fix $\varepsilon > 0$ small. Then

$$\varphi(r) - \varphi(\varepsilon) = \int_\varepsilon^r \varphi'(s)\, ds$$

ie

$$\fint_{\partial B(0,r)} g(x)\, dS(x) - \varphi(\varepsilon) = \frac{1}{n\alpha_n} \int_\varepsilon^r \frac{1}{s^{n-1}} \left( \int_{B(0,s)} f(x)\, dx \right) ds \quad (2)$$

Let $\psi(s) := \int_{B(0,s)} f(x)\,dx$

Then (tutorials / lectures) $\psi'(s) = \int_{\partial B(0,s)} f(x)\,dS(x)$.

Now integrate by parts (LHS of (2))

$$\int_\varepsilon^r \underbrace{\frac{1}{s^{n-1}}}_{= \frac{d}{ds}\left(\frac{-s^{2-n}}{n-2}\right)} \psi(s)\,ds = -\frac{1}{(n-2)} \underbrace{\left[\frac{1}{s^{n-2}}\psi(s)\right]_\varepsilon^r}_{A} + \frac{1}{(n-2)} \underbrace{\int_\varepsilon^r \psi'(s)/s^{n-2}\,ds}_{B}$$

[ $A = \frac{1}{r^{n-2}} \int_{B(0,r)} f(x)\,dx - \frac{1}{\varepsilon^{n-2}} \int_{B(0,\varepsilon)} f(x)\,dx$

$f$ continuous, so $\frac{1}{\varepsilon^n \alpha_n} \int_{B(0,\varepsilon)} f(x)\,dx \longrightarrow f(0)$ as $\varepsilon \downarrow 0$.

Hence $\frac{1}{\varepsilon^{n-2}} \int_{B(0,\varepsilon)} f(x)\,dx = \alpha_n \varepsilon^2 ⨍_{B(0,\varepsilon)} f(x)\,dx \longrightarrow 0$ as $\varepsilon \downarrow 0$.

$B = \int_\varepsilon^r \psi'(s)/s^{n-2}\,ds = \int_\varepsilon^r \frac{1}{s^{n-2}} \int_{\partial B(0,s)} f(x)\,dS(x) \Big)\,ds$

But for $x \in \partial B(0,s)$, $|x| = s$!

So $B = \int_\varepsilon^r \left( \frac{f(x)}{|x|^{n-2}}\,dS(x) \right) ds \overset{\text{(polar coords)}}{=} \int_{B(0,r)\setminus B(0,\varepsilon)} \frac{f(x)}{|x|^{n-2}}\,dx$

$$\xrightarrow{\varepsilon \downarrow 0} \int_{B(0,r)} \frac{f(x)}{|x|^{n-2}}\,dx$$

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Also, since $u$ is cont, $\varphi(\varepsilon) \to u(0)$ as $\varepsilon \downarrow 0$.

Hence, taking $\varepsilon \to 0$ in (2), we have

$$\fint_{\partial B(0,r)} g(x)\, dS(x) - u(0) =$$

$$\frac{1}{n\alpha_n}\left( \frac{-1}{(n-2)} \left( \frac{1}{r^{n-2}} \int_{B(0,r)} f(x)\, dx \right) + \frac{1}{(n-2)} \int_{B(0,r)} \frac{f(x)}{|x|^{n-2}}\, dx \right)$$

Rearrange to get result $\square$

(2) Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic. Let $O \in \mathbb{R}^{n\times n}$ be orthogonal.

Show

$$v(x) := u(Ox) \quad \text{is harmonic too.}$$

Write
$$O = (q_{ij}) = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & & & \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

$OO^t = O^tO = I$. So $\displaystyle\sum_{j=1}^n q_{ij} q_{kj} = \begin{cases} 1 & i=k \\ 0 & i \neq k. \end{cases}$

$\left( \text{and } \displaystyle\sum_{j=1}^n q_{ji} q_{jk} = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases} \right)$ rows (cols) orthonormal vectors in $\mathbb{R}^n$.

$$v(x) = u\left( \sum_{j=1}^n q_{1j} x_j, \sum_{j=1}^n q_{2j} x_j, \ldots, \sum_{j=1}^n q_{nj} x_j \right)$$

So $$\frac{\partial v}{\partial x_i}(x) = \sum_{k=1}^n q_{ki} \frac{\partial u}{\partial x_k}\Big( \underbrace{\textstyle\sum q_{1j}x_j, \ldots, \sum q_{nj}x_j}_{(=Ox)} \Big)$$

$$\frac{\partial^2 v}{\partial x_i^2} = \sum_{k=1}^n q_{ki} \underbrace{\frac{\partial}{\partial x_i} \big( u_{x_k}(Ox) \big)}$$

$$= \sum_{k=1}^n q_{ki} \sum_{m=1}^n q_{mi}\, u_{x_k x_m}(Ox)$$

Hence

$$\Delta v(x) = \sum_{i=1}^{n} v_{x_i x_i}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} q_{ki} \sum_{m=1}^{n} q_{mi}\, u_{x_k x_m}(Qx)$$

$$= \sum_{k=1}^{n} \sum_{m=1}^{n} \sum_{i=1}^{n} \underbrace{q_{ki} q_{mi}}_{=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}} u_{x_k x_m}(Qx)$$

$$= \sum_{k=1}^{n} u_{x_k x_k}(Qx) = \Delta u(Qx) = 0 \quad (u \text{ harmonic})$$

$\square$

(3) Suppose $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ has the form

$$u(x) = f(|x|) \quad \text{where} \quad f \in C^2(0,\infty)$$

Show

$$\Delta u(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|)$$

$$u(x) = f\left( (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \right)$$

$r = |x| \qquad \frac{\partial r}{\partial x_i} = \frac{1}{2}(2x_i)(x_1^2 + \ldots + x_n^2)^{-\frac{1}{2}}$

$= \frac{x_i}{|x|}$

So $\frac{\partial u}{\partial x_i}(x) = \frac{\partial r}{\partial x_i} f'(r) = \frac{x_i}{|x|} f'(|x|)$

chain rule

$$\frac{\partial^2 u}{\partial x_i^2}(x) = \frac{x_i}{|x|} \frac{\partial}{\partial x_i}\left( f'(|x|) \right) + f'(|x|) \frac{\partial}{\partial x_i}\left( \frac{x_i}{|x|} \right)$$

$$\frac{\partial}{\partial x_i}\left( f'(|x|) \right) = \frac{x_i}{|x|} f''(|x|) \quad (\text{as before})$$

$$\frac{\partial}{\partial x_i}\left( \frac{x_i}{|x|} \right) = \frac{\partial}{\partial x_i}\left( x_i (x_1^2 + \ldots + x_n^2)^{-\frac{1}{2}} \right)$$

chain rule

$$= x_i \left(-\tfrac{1}{2}\right)(2x_i)(x_1^2 + \ldots + x_n^2)^{-\frac{3}{2}} + (x_1^2 + \ldots + x_n^2)^{-\frac{1}{2}}$$

$$= \frac{-x_i^2}{|x|^3} + \frac{1}{|x|}$$

Here we have

$$\frac{\partial^2 u}{\partial x_i^2}(x) = \frac{x_i^2}{|x|^2} f''(|x|) + \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right) f'(|x|)$$

$\sum x_i^2 = |x|^2$ !

So

$$\Delta u(x) = \sum_{i=1}^{n} \frac{x_i^2}{|x|^2} f''(|x|) + \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\right) f'(|x|)$$

$$= \frac{|x|^2}{|x|^2} f''(|x|) + \frac{n}{|x|} f'(|x|) - \frac{|x|^2}{|x|^3} f'(|x|)$$

$$= f''(|x|) + \frac{(n-1)}{|x|} f'(|x|)$$

//

If $u$ has this form (it's rotationally invariant) and $\Delta u = 0$, then writing $r = |x|$,

$$f''(r) + \frac{n-1}{r} f'(r) = 0$$ $1^{st}$ order ODE in $f'$

Solve to find $f'$: $g = f'$.

$$\frac{g'(r)}{g(r)} = -\frac{(n-1)}{r}$$

Integrate ~~$\ln g(r) + C =$~~

$$\int_1^r \frac{g'(s)}{g(s)} ds = -(n-1)\int_1^r \frac{1}{s} ds$$

$$\ln g(r) = -(n-1)\ln r + C = \ln r^{-(n-1)} + C$$

$$g(r) = C \cdot r^{-(n-1)} = f'(r)$$

So $f(r) = c_1 r^{-(n-2)} + c_2$

$u$ must have the form $u(x) = \dfrac{c_1}{|x|^{n-2}} + c_2$ . //

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